A NEW FORMULATION FOR REFLECTION OF A POINT IN A PARAMETRIC CURVED MIRROR

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Abstract

In this paper, by generalizing a work of Shikhar Mittal for deriving the equation of the curve obtained on reflection of a point object in a curved mirror (in the form $y = f(x)$), we compute the intersections of reflected rays from a curved mirror in the parametric form γ(t)=(x(t),y(t)) and present some different examples which are caculated by Maple. The results can be used in computer graphics for ray tracing and in geometrical optics for finding the images of a point in such mirrors and in the theory of waves for discribing the reflction of a spherical wavefront.

Key words: Geometrical Optics, Parametric Curved Mirror, Intersection of Reflected Rays.

1. Introduction

In geometrical optics which can be considered as the study of the phenomenon of refraction and reflection, it is convenient to investigate the behavior of rays or directed lines in the plane (see for example [3, 4, 5, 6]). According to the famous law due to Snell (which as it is mentioned in materiaislamica.com or in wikipedia, has been first accurately described by the Persian scientist Ibn Sahl at the Baghdad court in 984), the angle of incidence is equal to the angle of reflection and it is well known that the incident ray, normal and the reflected ray lie in the same plane. It means that if S is a point on a ray which strikes the reflector at point P and if N is the normal line to the reflector at P, then there is a point C on the reflected ray from point P that is symmetrical point of S with respect to N.

In [1] Bhattacharjee introduced the generalized vector laws of reflection and refraction. By this new formulation he described in [2] the cases of reflection of a plane wavefront of light by plane as well as spherical reflecting surfaces.

Mittal in [7] has derived the equation of the curve obtained on reflection of a point object in a curved mirror (in the form $y=f(x)$) if the object and the mirror are placed on the 2D Cartesian plane. Here we generalize the results of [7] for a mirror defined by a parametric curve $\gamma(t) = (x(t), y(t))$ and present a new formulation and different examples that should be calculated by this new formulation. Although we obtain the results for light rays, sometimes a mechanical wavefront can be considered as a family of wave vectors and the behavior of light rays can be used for these situations, e.g. a spherical wavefront may be considered as a union of wave directions which come from a source point. But there are some differents between these two cases. It is well known that the image of a point object may be in front or at behind of the mirror, therfore in some of examples the curve of intersection point of reflected rays is at the behind of the mirror. This phenomena, however, will not happen for the reflection of a mechanical wavefront and the reflected wavefront will be in front of the reflecting surface. Although our formulation can be used for this viewpoint, but here we put aside the details of the situation for mechanical waves.

2. The image of a point in a parametric curved mirror

In order to finding the image (or images) of a point in a parametric curved mirror $\gamma(t) = (x(t), y(t))$, at first we should describe the generalized equation of the family of reflected rays on such mirrors. Then we recall the method that introduced by Mittal in [7] for finding the intersection of a family of lines, and by which we will generalize the formula of the intersection point of reflected rays from the mirror γ. In this way, two theorems are proved and the relations are simplified using determinants and inner products.

There are some examples that cannot be described by the previous formula in [7] and should be computed by this new formulation. Some of these examples are shown in subsection 2.4.

2.1. Equation of the family of reflected rays

Now suppose that we have a curved mirror defined by parametric equation $\gamma(t) = (x(t), y(t))$ in $2D$ Cartesian plane. Let $S = (a, b)$ be a point object in this plane that as a source sends a family of rays to the mirror. We assume that the mirror is smooth and perfectly reflecting so that refraction, absorption or dispersion of beam itself does not occur. We want to find the image (or images) of S in this mirror by a generalization of the method described in [7].

At first, we need some notations which help us to make formulas short and clear. Let $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and w be 3 vectors in the plane, we set

$$|u, v| = u_1 v_2 - u_2 v_1 = \det(u, v), \quad \langle u, v\rangle = u_1 v_1 + u_2 v_2, \quad U_{(u,v,w)} = \begin{pmatrix} |u, v| \\ \langle u, w\rangle \end{pmatrix}. \tag{1}$$

Now let $P = (x(t), y(t))$ be a point on the curve $\gamma(t)$. It is well known that the equation of normal line N at point P to the curve γ is $y - y(t) = -\frac{\dot{x}(t)}{\dot{y}(t)}(x - x(t))$, in which $\dot{x}$ and $\dot{y}$ show respectively the differentiation of x and y with respect to t.

Consider an incident ray from point S in the plane that strikes to point P on the curve $\gamma(t)$. We should find the equation of line which is coincides with the reflected ray from P. Let C be the symmetrical point to S with respect to the normal N at point P on γ.

The equation of the line passing through points S and C that is perpendicular to N and hence is parallel to the tangent line to γ at point P is $y - b = \frac{\dot{y}(t)}{\dot{x}(t)}(x - a)$.

In order to finding the coordinates of the point C we use the coordinates of the mid point M of the line segment SC that is the intersection of two lines N and T. By some calculations we obtain

$$x_M = \frac{\dot{x}(t)\langle\dot{\gamma}(t), \gamma(t)\rangle - \dot{y}(t).\det(\dot{\gamma}(t), S)}{|\dot{\gamma}(t)|^2}, \qquad y_M = \frac{\dot{y}(t)\langle\dot{\gamma}(t), \gamma(t)\rangle + \dot{x}(t).\det(\dot{\gamma}(t), S)}{|\dot{\gamma}(t)|^2} \tag{2}$$

in which $|\dot{\gamma}(t)|^2 = \dot{x}^2(t) + \dot{y}^2(t)$. Using notations in (1) we have

$$x_M = \frac{1}{|\dot{\gamma}(t)|^2}\left|\dot{\gamma}(t), U_{(\dot{\gamma},S,\gamma)}\right|, \qquad y_M = \frac{1}{|\dot{\gamma}(t)|^2}\langle\dot{\gamma}(t), U_{(\dot{\gamma},,S,\gamma)}\rangle \tag{3}$$

Since M is the mid point of the line segment SC so

$$x_C = 2x_M - a, \qquad y_C = 2y_M - b. \tag{4}$$

Hence

$$x_C = \frac{\dot{x}(t)\langle\dot{\gamma}(t), 2\gamma(t) - S\rangle - \dot{y}(t)\det(\dot{\gamma}(t), S)}{|\dot{\gamma}(t)|^2}, \qquad y_C = \frac{\dot{y}(t)\langle\dot{\gamma}(t), 2\gamma(t) - S\rangle + \dot{x}(t)\det(\dot{\gamma}(t), S)}{|\dot{\gamma}(t)|^2} \tag{5}$$

which by (1) can be written in the following shorter form

$$x_C = \frac{1}{|\dot{\gamma}(t)|^2}\left|\dot{\gamma}(t), U_{(\dot{\gamma},S,2\gamma-S)}\right|, \qquad y_C = \frac{1}{|\dot{\gamma}(t)|^2}\langle\dot{\gamma}(t), U_{(\dot{\gamma},S,2\gamma-S)}\rangle. \tag{6}$$

Now having two points P and C of the line $\overleftrightarrow{PC}$ we can obtain the equation of this line as $y = mx + c$ in which m and c, when we set $\Gamma = \begin{pmatrix} |(\gamma(t) - S), \dot{\gamma}(t)| \\ \langle(\gamma(t) - S), \dot{\gamma}(t)\rangle \end{pmatrix}$ and $\Gamma_1 = \begin{pmatrix} |\dot{\gamma}(t), \gamma(t)| \\ \langle\dot{\gamma}(t), \gamma(t)\rangle \end{pmatrix}$, are as follows

$$m = m(t) = \frac{y_C(t) - y(t)}{x_C(t) - x(t)} = \frac{\dot{y}(t)\langle(\gamma(t) - S), \dot{\gamma}(t)\rangle + \dot{x}(t)\det\big((\gamma(t) - S), \dot{\gamma}(t)\big)}{\dot{x}(t)\langle(\gamma(t) - S), \dot{\gamma}(t)\rangle - \dot{y}(t)\det\big((\gamma(t) - S), \dot{\gamma}(t)\big)} = \frac{\langle\dot{\gamma}(t), \Gamma\rangle}{|\dot{\gamma}(t), \Gamma|} \tag{7}$$

and

$$c = c(t) = y(t) - mx(t) = \frac{\langle(\gamma(t) - S), \dot{\gamma}(t)\rangle\det\big(\dot{\gamma}(t), \gamma(t)\big) - \langle\dot{\gamma}(t), \gamma(t)\rangle\det\big((\gamma(t) - S), \dot{\gamma}(t)\big)}{\dot{x}(t)\langle\dot{\gamma}(t), (\gamma(t) - S)\rangle - \dot{y}(t)\det\big((\gamma(t) - S), \dot{\gamma}(t)\big)} = \frac{|\Gamma_1, \Gamma|}{|\dot{\gamma}(t), \Gamma|} \tag{8}$$

Thus it is proved that

Theorem 1. By the above formulation when the point P and so the normal N change on the curve $\gamma(t)$, we have the family of reflected rays from mirror $\gamma(t)$ with the equation

$$y = m(t)x + c(t), \quad m(t) = \frac{\langle \dot{\gamma}(t), \Gamma \rangle}{|\dot{\gamma}(t), \Gamma|}, \quad c(t) = \frac{|\Gamma_1, \Gamma|}{|\dot{\gamma}(t), \Gamma|}.$$

2.2. Intersection of family of lines

As it is described in [7], if a parametric family of lines

$$y = m(t)x + c(t) \tag{9}$$

with parameter $t \in \mathbb{R}$ is given, and the point $(x_I(t), y_I(t))$ is the intersection of two infinitesimally apart lines $y = mx + c$ and $y = (m + dm)x + (c + dc)$, then $y_I(t) = mx_I(t) + c$ in which

$$x_I(t) = -\frac{\dot{c}(t)}{\dot{m}(t)}. \tag{10}$$

Now we can generalize the discussion in [7]. Suppose that the family (9) are defined by a parametric curve $\gamma(t) = (x(t), y(t))$. Then we have $c(t) = y(t) - m\, x(t)$, as in (8). Substituting this in (10) we have

$$x_I(t) = x(t) - \frac{\dot{y}(t) - m(t)\dot{x}(t)}{\dot{m}(t)}. \tag{11}$$

Moreover

$$y_I(t) = c(t) + mx_I(t). \tag{12}$$

2.3. Intersection point of reflected rays from γ

We know that for finding the coordinates of the intersection of lines that are reflected rays from the parametric curve $\gamma(t)$, it is necessary to obtain $\dot{m}(t)$ by the equation (7), that is after some calculations as follows

$$\dot{m}(t) = \frac{|\dot{\gamma}(t)|^2 \left(2|(\gamma(t)-S)|^2\, |\dot{\gamma}(t), \ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2\, |(\gamma(t)-S), \dot{\gamma}(t)|\right)}{\left(\dot{x}(t)\langle \dot{\gamma}(t), (\gamma(t)-S)\rangle - \dot{y}(t)\, |(\gamma(t)-S), \dot{\gamma}(t)|\right)^2} \tag{13}$$

or simply $\dot{m}(t) = |\dot{\gamma}(t)|^2 \frac{|\Gamma_2, \Gamma_3|}{|\dot{\gamma}(t), \Gamma|^2}$, in which $\Gamma_2 = \begin{pmatrix} |\dot{\gamma}(t), \ddot{\gamma}(t)| \\ \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle \end{pmatrix}$ and $\Gamma_3 = \begin{pmatrix} |\gamma(t) - S, \dot{\gamma}(t)| \\ 2\langle \gamma(t) - S, \gamma(t) - S \rangle \end{pmatrix}$.

Now by substituting equation (13) in equation (11) we can obtain the x-coordinate of the intersection point of reflected rays, namely

$$x_I(t) = x(t) - \frac{|\dot{\gamma}(t), (\gamma(t)-S)| \left(\dot{x}(t)\langle (\gamma(t)-S), \dot{\gamma}(t)\rangle - \dot{y}(t)\, |(\gamma(t)-S), \dot{\gamma}(t)|\right)}{2|(\gamma(t)-S)|^2\, |\dot{\gamma}(t), \ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2\, |(\gamma(t)-S), \dot{\gamma}(t)|} \tag{14}$$

that has a shorter form $x_I(t) = x(t) - |\dot{\gamma}(t), \gamma(t) - S| \frac{|\dot{\gamma}(t), \Gamma|}{|\Gamma_2, \Gamma_3|}$.

By replacing (14) in (12) we find y-coordinate of intersection point of reflected rays that is

$$y_I(t) = y(t) - \frac{|\dot{\gamma}(t), (\gamma(t)-S)| \left(\dot{y}(t)\langle (\gamma(t)-S), \dot{\gamma}(t)\rangle + \dot{x}(t)\, |(\gamma(t)-S), \dot{\gamma}(t)|\right)}{2|(\gamma(t)-S)|^2\, |\dot{\gamma}(t), \ddot{\gamma}(t)| - |\dot{\gamma}(t)|^2\, |(\gamma(t)-S), \dot{\gamma}(t)|} \tag{15}$$

whose smaller form will be $y_I(t) = y(t) - |\dot{\gamma}(t), \gamma(t) - S| \frac{\langle \dot{\gamma}(t), \Gamma \rangle}{|\Gamma_2, \Gamma_3|}$. Therefore it is shown that

Theorem 2. The intersection point of reflected rays from the mirror $\gamma(t)$ by the above notations has the following coordinates

$$x_I(t) = x(t) - |\dot{\gamma}(t), \gamma(t) - S| \frac{|\dot{\gamma}(t), \Gamma|}{|\Gamma_2, \Gamma_3|}, \qquad y_I(t) = y(t) - |\dot{\gamma}(t), \gamma(t) - S| \frac{\langle \dot{\gamma}(t), \Gamma \rangle}{|\Gamma_2, \Gamma_3|}$$

that gives the parametric curve of the image of the source point S.

2.4. Some examples

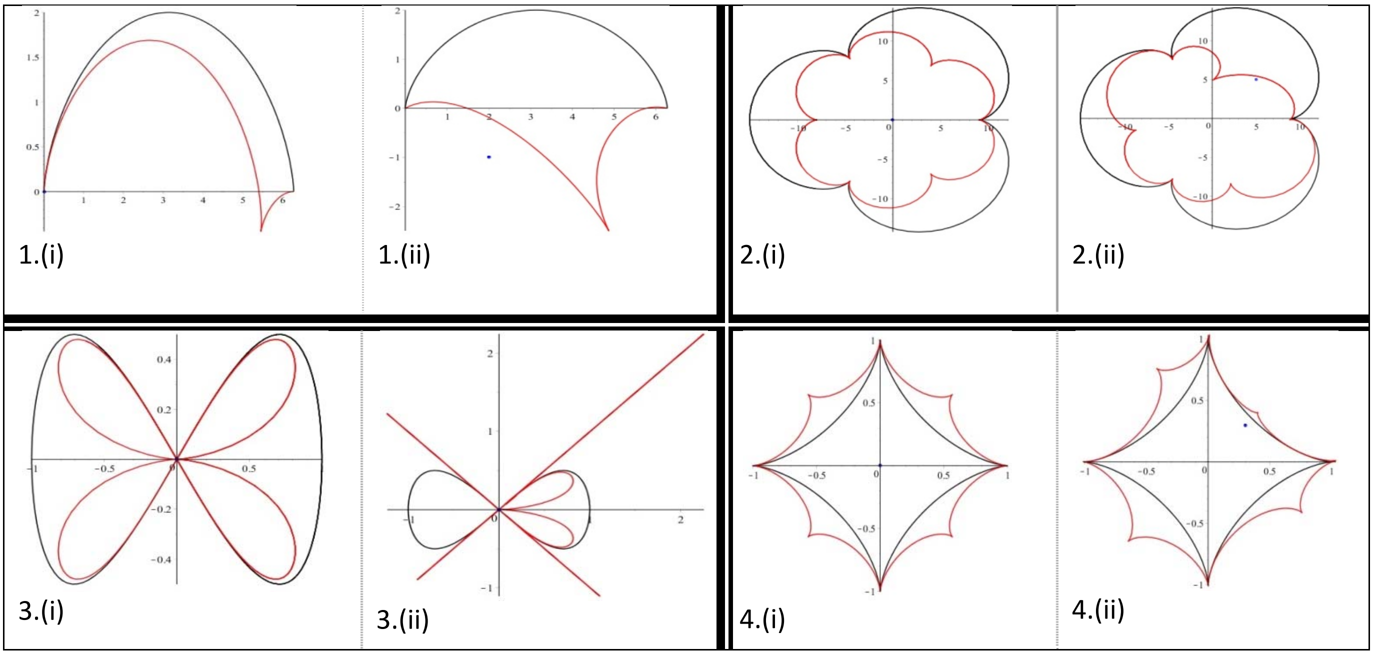
Here we present some examples from different parametric curves as the mirror or the reflector curve when locus of the point object (or the source point) S can be changed. Note that in each of examples the red (or in printed paper, gray) color curve is the curve of images of S. The blue dot is the source point S. Although, by changing the point S for a fixed curved mirror we can illustrate many different shapes of the images curve, but here we illustrate only two of them in each example. In Figures 1.(i) and 1.(ii), the curve $\gamma(t) = (t - \mathrm{sin}t, 1 - \mathrm{cos}t)$ is a Cycloid, $t \in [0, 2\pi]$, $S = (0,0)$ and $S = (2, -1)$, respectively.

When the reflector is defined by a closed curve, it is clear that the light rays will be reflected many times. In the following we will show only the first intersection of the reflected rays.

In Figures 2.(i) and 2.(ii), $\gamma(t) = (12\mathrm{cos}t - 3\cos(4t), 12\mathrm{sin}t - 3\sin(4t))$ is an Epicycloid, $t \in [0,2\pi]$, S=(0,0) and S=(5,5). It is shown that when the locus of S is changed, how the curve of images is deformed.

In Figures 3.(i) and 3.(ii), $\gamma(t) = (\mathrm{sin}t, \mathrm{sin}t\mathrm{cos}t)$ is the Eight curve, $t \in [0,2\pi]$, with $S = (0,0)$ and $S = (0.0001,0)$. It should be mentioned that in Figure 3.(ii) we have two crossed line segments, i.e. the image is bounded.

In Figures 4.(i) and 4.(ii), $\gamma(t) = \left(\cos^3(t), \sin^3(t)\right)$ is the Astroid, $t \in [0,2\pi]$, for which it is shown the change of the first intersection point of reflected rays when S moves from origin to point (0.3,0.3).



3. Conclusions

When a point object $S = (a, b)$ is in front of a parametric curved mirror $\gamma(t) = (x(t), y(t))$ in $2D$ Cartesian plane and as a source sends a family of rays to the mirror γ, it is shown that the intersections of the reflected rays from γ can be described as a parametric curve in the plane. It is a generalization of the case when the equation of the mirror is $y = f(x)$, which introduced in [7].

References

[1] P.R. Bhattacharjee, The generalized vectorial laws of reflection and refraction. *Eur. J. Phys.*, 26(5), 901–911, 2005.

[2] P.R. Bhattacharjee, On finding the correlation between the incident wavefront, reflected (refracted) wavefront and the reflecting surface (surface of discontinuity). *Optik*, 127, 4819–4823, 2016.

[3] C. Dupin, Applications de la geometrie *Memoire presente a lAcademie des Sciences en 1816*, publie a Paris en 1822.

[4] W. R. Hamilton, Theory of systems of rays. *Part First and Part Second (1827). Part first: Trans. Royal Irish Academy, 15 (1828), pp. 69 – 174. Part Second: manuscript. In Sir William Rowan Hamilton mathematical Works, vol. I, chapter I, Cambridge University Press, London, 1931*.

[5] E. L. Malus, *Journal de lEcole Polytechnique*, 7, pp. 1–44 and 84–129, 1808.

[6] Charles-Michel Marle, A direct proof of Malus' theorem using the symplectic structure of the set of oriented straight lines. *http://arxiv.org/abs/1409.0381v1*, 1 Sep 2014.

[7] Shikhar Mittal, Reflection of a Point Object in an Arbitrary Curved Mirror. *http://arxiv.org/abs/1903.01074v1*, 4 March 2019.